

On the energy translation invariance of probability distributions

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We comment on the problem of energy translation invariance of probability distribution and present some observations. It is shown that a probability distribution can be invariant in the thermodynamic limit if there is no long term interaction or correlation and no relativistic effect. So this invariance should not be considered as a universal theoretical property. Some peculiarities within the invariant q -exponential distribution reveal that the connection of the current nonextensive statistical mechanics to thermodynamics might be disturbed by this invariance.

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I. INTRODUCTION

The nonextensive statistical mechanics (NSM) [1–7] is a generalization of the Boltzmann-Gibbs statistics (BGS) intended to study complex nonlinear systems having long-range correlations and fractal or chaotic state space for which BGS is no more valid. Due to the generalized character of this theory, the scientists of this field have been brought back to reflections about many fundamental aspects of physics theory. Can we keep the additivity of, e.g., entropy, energy and volume for complex systems? What are the possible relation between the probabilities of interacting systems? What is thermodynamic equilibrium in this case? How to define it for systems of same nonextensive nature and for systems of different nonextensive nature? Is the classical probability definition still valid in chaotic or fractal phase space? Does the completeness of information about systems containing independent parts still apply to systems with correlated or overlapped parts ... ? All these questions concerning the foundation of NSM are far from being answered and it would be too soon to conclude. A complete comprehension of these aspects is necessary to finally determine the validity limites of NSM and to know the kind of systems to which NSM should be applied.

One of the most worrying problems of the NSM distribution given by

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$$p_i = \frac{1}{Z} [1 - (1 - q)\beta e_i]^{\frac{1}{1-q}} \quad (1)$$

is that it is not invariant under uniform translation of the energy spectra e_i [3,8]. Only when $q = 1$, Eq.(1) becomes $p_i = \frac{1}{Z} e^{-\beta e_i}$ and the invariance can be recovered. This q -exponential distribution can be given by maximizing Tsallis entropy¹ with the unnormalized expectation [2]

$$U = \sum_i^w e_i p_i^q. \quad (2)$$

and the normalization $\sum_i^w p_i = 1$ from which the partition function Z is calculated. Eq.(1) can also be found, with a different partition function, by maximizing Tsallis entropy with Eq.(2) and an anomalous normalization $\sum_i^w p_i^q = 1$ [5]. β can be identified to the inverse temperature within a non-additive energy scenario [5–7,9] or with an additive energy approximation [10,11]).

From the power law distribution Eq.(1), it is obvious that, if we replace e_i by $e_i + C$ where C is constant, we find $p_i(e_i + C) \neq p_i(e_i)$, excepted that $q = 1$. If p_i varies with C , then the variance of all the thermodynamic functions will be disturbed by C . For example, for the energy U , we have

$$\begin{aligned} U(e + C) &= \sum_i (e_i + C) p_i^q(e_i + C) \\ &= \frac{1}{Z^q(e + C)} \sum_i (e_i + C) [1 - (1 - q)\beta(e_i + C)]^{\frac{q}{1-q}} \\ &= \frac{-1}{Z^q(e + C)} \frac{\partial}{\partial \beta} \sum_i [1 - (1 - q)\beta(e_i + C)]^{\frac{1}{1-q}} \\ &= \frac{-1}{Z^q(e + C)} \frac{\partial Z(e + C)}{\partial \beta} \\ &= -\frac{\partial}{\partial \beta} \frac{Z^{1-q}(e + C) - 1}{1 - q} \\ &\neq U + C \end{aligned} \quad (3)$$

where $U = -\frac{\partial}{\partial \beta} \frac{Z^{1-q}-1}{1-q}$ is the unnormalized expectation without energy translation ($C = 0$). This theoretical feature seems difficult to accept according

¹Tsallis entropy is given by $S = -k \frac{\sum_{i=1}^w p_i - \sum_{i=1}^w p_i^q}{1-q}$, ($q \in R$) [1]

to some scientists [3,8]. Later, a normalized expectation E of energy was proposed [3,4] on the basis of the so called escort probability

$$E = \frac{\sum_i^w p_i^q e_i}{\sum_i^w p_i^q}. \quad (4)$$

which, introduced into the maximization of Tsallis entropy as a constraint replacing Eq.(2), allows to obtain

$$p_i = \frac{[1 - (1 - q)\beta(e_i - E)]^{\frac{1}{1-q}}}{Z} \quad (5)$$

with

$$Z = \sum_i^w [1 - (1 - q)\beta(e_i - E)]^{\frac{1}{1-q}}. \quad (6)$$

The distribution Eq.(5) is invariant under energy translation because, according to reference [3], if we add a constant to e_i , we have the same constant added to E . Recently, it is argued [12] that, for a correct and successful maximization of Tsallis entropy leading to Eq.(5), the expectation Eq.(4) as a constraint should be replaced by the unnormalized expectation Eq.(2) minus the expectation given by Eq.(4), i.e. $U - \sum_i p^q E = \sum_i p^q (e_i - E)$.

In this paper, we will firstly discuss different conditions for the validity of energy translation invariance of probability distributions. Then we present some observations about the relevant problems or theoretical peculiarities within invariant NSM. The probability invariance with a generalized energy shift consistent with NSM framework is discussed.

II. ENERGY TRANSLATION INVARIANCE OF MAXWELL-BOLTZMANN DISTRIBUTION

As well known, BGS distribution is invariant under energy translation. This property conforms with the intuition that the particle distributions in a container will not be disturbed when, for example, you raise the container to a higher point or move it at a different speed. This invariance of BGS is directly related to that of classical mechanics which leaves the interaction potential completely arbitrary and so is invariant if we add a constant to the total energy. In addition, the exponential form of BGS allows following invariant property :

$$p(e_i) = \frac{1}{Z} e^{-\beta e_i} = \frac{e^{-\beta C}}{e^{-\beta C}} \frac{e^{-\beta e_i}}{\sum_j e^{-\beta e_j}} = \frac{e^{-\beta(e_i+C)}}{\sum_j e^{-\beta(e_j+C)}} = p(e_i + C) \quad (7)$$

So p_i will not be changed if we add a constant to the energy spectrum e_i . The partition function will be changed as follows :

$$Z(e + C) = \sum_j e^{-\beta(e_j+C)} = e^{-\beta C} \sum_j e^{-\beta e_j} = e^{-\beta C} Z(e) \quad (8)$$

which gives following internal energy U :

$$U(e + C) = -\frac{\partial}{\partial \beta} \ln Z(e + C) = -\frac{\partial}{\partial \beta} \ln Z(e) + C = U(e) + C. \quad (9)$$

It is instructive to see the role of thermodynamic limit in this invariance when we study the velocity distribution of ideal gas. Suppose that \vec{v}_i is the velocity of the i^{th} particle of mass m with respect to the container or to the center of gravity of the gas. The total kinetic energy of the gas is given by

$$e = \sum_l \frac{1}{2} m \vec{v}_l^2. \quad (10)$$

Now we are in another system of reference in which the container moves at a velocity \vec{v}_0 , we have

$$e' = \sum_l \frac{1}{2} m (\vec{v}_0 + \vec{v}_l)^2 = \frac{1}{2} m (\sum_l \vec{v}_l^2 + \sum_l \vec{v}_0^2 + 2 \sum_l \vec{v}_0 \cdot \vec{v}_l). \quad (11)$$

In the thermodynamic limit, the third term at the right hand side of Eq.(11) is null because $\sum_{l=1}^N \vec{v}_l = 0$ if $N \rightarrow \infty$. We then have $e' = \frac{1}{2} m \sum_l \vec{v}_l^2 + \frac{1}{2} m N \vec{v}_0^2 = e + \frac{1}{2} M \vec{v}_0^2$ where M is the total mass of the gas. Put this in Eq.(7) in replacing the summation by an integration in v_l , we find that the distribution law does not change with v_0 .

Briefly, according to the above discussion, we have at least three conditions on the invariance of the distribution.

1. The mechanics basis must be non relativist, i.e. an arbitrary constant can be added to the energy.
2. Thermodynamic limit holds, i.e. $N \rightarrow \infty$.
3. The distribution law must be exponential.

It is well known that the third condition needs the second one and, in addition, the harsh assumptions that the interaction or correlation in the system is negligible or of short term so that the different parts of the system under consideration are independent and additive. In view of the above mentioned conditions, it is difficult to say that the energy translation invariance of probability distribution is an universal theoretical property. Indeed, if we consider for example the relativistic effect, we can no more add constant to the energy e due to its relation with the total mass M given by

$$e = Mc^2 \quad (12)$$

where c is the light speed. e is not arbitrary because M can not be changed arbitrarily. If we suppose a system composed of many elements of mass m_i and energy ϵ_i ($i = 1, 2, 3, \dots$) in interaction with total potential energy V , M will be given by :

$$M = \sum_i m_i + V/c^2 \quad (13)$$

or

$$Mc^2 = \sum_i m_i c^2 + V = \sum_i \epsilon_i + V \quad (14)$$

with $\epsilon_i = m_i c^2$ for i^{th} element. It is obvious that M , ϵ_i and V cannot be changed, or the variance of the theory would be perturbed [13].

On the other hand, the relativistic kinetic energy of particles ($mc^2/\sqrt{1-v_l^2/c^2}$) is not linear function of v_l^2 . So what we did in Eq.(11) leading to $e' = e + mv_0^2/2$ does not hold for relativistic gas. This means that the relativistic velocity distribution will change with \vec{v}_0 even with exponential distributions.

Now if we consider complex systems with long range interactions which are not limited between the walls of the containers, the things are more complex than with short range interactions because the entropy and energy can be nonadditive so that the exponential distribution does not exist any more. So the invariant distribution is only a theoretical property in special cases. It should not be claimed for general cases. It is worth emphasizing that, up to now, all the successful applications of the distribution Eq.(1) are never disturbed by the fact that it is not invariant with energy translation. On the contrary, the invariant distributions like Eq.(5) were shown to present some serious theoretical difficulties, as discussed below.

III. ENERGY TRANSLATION INVARIANCE OF NONEXTENSIVE DISTRIBUTION

Nevertheless, there is up to now no any explicit reason to completely exclude invariant nonextensive distribution which is recently discussed in a general way and claimed to be an universal property verified by any thermostatics with linear or normalized expectation, whatever the entropy form [8]. We have shown in the previous section that this conclusion is not true. But a question remains open : is it possible to obtain the invariance with a distribution function like Eq.(5)? In this section, we would like to show some observations about Eq.(5) and discuss some of its theoretical peculiarities.

If we want a distribution p_i to have energy translation invariance, we have to first of all define an invariant normalized expectation satisfying $U(e + C) = U(e) + C$. This is possible with the expectation Eq.(4) (or Eq.(2) plus the anomalous normalization $\sum_i^w p_i^q = 1$ [5–7]) under the assumption that $p_i(e_i + C) = p_i(e_i)$. That is

$$\begin{aligned} E(e + C) &= \sum_i (e_i + C) p_i^q(e_i + C) / \sum_i p_i^q(e_i + C) \\ &= \sum_i (e_i + C) p_i^q / \sum_i p_i^q \\ &= E + C \end{aligned} \quad (15)$$

Then the method of [12] can be adapted to introduce $\sum_i^w p_i^q E$ as the invariance constraint into the following functional

$$A = -\frac{\sum_{i=1}^w p_i - \sum_{i=1}^w p_i^q}{1 - q} - \alpha \sum_i^w p_i - \beta \sum_i^w p_i^q e_i + \gamma \sum_i^w p_i^q E \quad (16)$$

Let $\frac{\partial A}{\partial p_i} = 0$, we obtain the following distribution :

$$p_i = \frac{[1 - (1 - q)(\beta e_i - \gamma E)]^{\frac{1}{1-q}}}{Z} \quad (17)$$

For p_i to be invariant under energy translation, considering Eq.(15), we have to set $\beta = \gamma$, which leads to the invariant distribution Eq.(5).

IV. WHAT IS WRONG WITH THE INVARIANT DISTRIBUTION?

From the invariant distribution Eq.(5), we easily show that

$$Z^{1-q} \sum_i p_i = \sum_i p_i^q [1 - (1-q)\beta(e_i - E)] \quad (18)$$

Due to the invariant factor $(e_i - E)$, Eq.(18) leads to

$$\sum_i p_i^q = Z^{1-q} \quad (19)$$

and

$$Z = \sum_i [1 - (1-q)\beta(e_i - E)]^{\frac{q}{1-q}}. \quad (20)$$

or

$$\sum_i [1 - (1-q)\beta(e_i - E)]^{\frac{1}{1-q}} = \sum_i [1 - (1-q)\beta(e_i - E)]^{\frac{q}{1-q}}. \quad (21)$$

Eqs.(19) and Eq.(21) are obviously basic relations of the invariant theory and should hold for arbitrary value of q , β and e_i . We will see that most of the problems encountered below are intrinsically related to these two equalities.

1. Let us begin by addressing the problem of the calculation of nonextensive term or correlation in energy (or any other quantities of interest) with Eq.(5).

Suppose an isolated system C composed of two subsystems A and B in thermal equilibrium. It was shown [14,7] that, with Tsallis entropy, the equilibrium condition yields for even interacting subsystems the following product probability law

$$p_{ij}(C) = p_i(A)p_j(B) \quad (22)$$

and entropy pseudoadditivity

$$S(A+B) = S(A) + S(B) + \frac{1-q}{k} S(A)S(B). \quad (23)$$

For NSM with q -exponential distribution, the product probability means that A and B are correlated and should give the pseudoadditivity associated with the quantity of interest. But from Eqs. (5) and (22), we straightforwardly obtain :

$$\begin{aligned} e_{ij}(A+B) - E(A+B) &= [e_i(A) - E(A)] + [e_j(B) - E(B)] \\ &+ (q-1)\beta[e_i(A) - E(A)][e_j(B) - E(B)]. \end{aligned} \quad (24)$$

Without additional hypothesis, this equality does not lead to any explicit relation between the total energy and the energies of the subsystems A and B , which is absolutely necessary for defining temperature, pressure and chemical potential as the measures of equilibrium.

Recently, some authors [10,11] proposed neglecting the nonextensive term in energy and writing

$$e_{ij}(A + B) = e_i(A) + e_j(B) \quad (25)$$

and

$$E(A + B) = E(A) + E(B). \quad (26)$$

We have seen that this additive energy approximation allows to reconcile the invariant theory with the zeroth law of thermodynamic [10,11] and to establish a generalized thermodynamics. But it should not be forgotten that this is only an approximate approach. In other words, the invariant theory does not have vigorously defined temperature. This is one of its intrinsic flaws.

2. Now we address a mathematical problem related to Eqs.(19) and (21). From Eq. (19), Tsallis entropy can be recast as

$$S = k \frac{Z^{1-q} - 1}{1 - q}. \quad (27)$$

Then we calculate the following derivative :

$$\frac{dS}{dE} = \frac{k}{Z^q} \frac{dZ}{dE}. \quad (28)$$

First we take the Z given by Eq. (6), we obtain :

$$\begin{aligned} \frac{dS}{dE} &= \frac{k}{Z^q} \frac{d}{dE} \sum_i^w [1 - (1 - q)\beta(e_i - E)]^{\frac{1}{1-q}} \\ &= \frac{k\beta}{Z^q} \sum_i^w [1 - (1 - q)\beta(e_i - E)]^{\frac{q}{1-q}} \\ &= k\beta Z^{1-q}. \end{aligned} \quad (29)$$

But considering Eq.(21), we can also take the Z of Eq. (20), this time we obtain

$$\frac{dS}{dE} = \frac{qk\beta}{Z^q} \sum_i^w [1 - (1-q)\beta(e_i - U)]^{\frac{2q-1}{1-q}}. \quad (30)$$

Comparing Eq.(30) to Eq.(29), we get

$$Z = q \sum_i^w [1 - (1-q)\beta(e_i - E)]^{\frac{2q-1}{1-q}}. \quad (31)$$

If we put Eq. (31) into Eq. (28) and continue in this way for n times, we will find

$$\begin{aligned} Z &= \sum_i^w [1 - (1-q)\beta(e_i - E)]^{\frac{q}{1-q}} \\ &= q \sum_i^w [1 - (1-q)\beta(e_i - E)]^{\frac{2q-1}{1-q}} \\ &= q(2q-1) \sum_i^w [1 - (1-q)\beta(e_i - E)]^{\frac{3q-2}{1-q}} \\ &= q(2q-1)(3q-2) \sum_i^w [1 - (1-q)\beta(e_i - E)]^{\frac{4q-3}{1-q}} \\ &= q(2q-1)(3q-2)\dots(nq-n+1) \sum_i^w [1 - (1-q)\beta(e_i - E)]^{\frac{(n+1)q-n}{1-q}} \end{aligned} \quad (32)$$

with $n = 0, 1, 2, \dots$. We create in this way a series of equalities which seem not to hold. For example, if we take the second equality of Eq. (32) and let $q \rightarrow 0$, the right-hand side will tend to zero and the left-hand side to $\sum_i^w 1 = w$. The result is $w \rightarrow 0$. This same result can also be obtained for $q \rightarrow \frac{1}{2}$ if we take the third equality of Eq. (32) and for $q \rightarrow \frac{2}{3}$ with the forth equality and so on. These singular points in q value do not conform with the hypothesis that Eq. (21) is a basic relation of the theory. It seems to us that these equalities are valid only when $q \rightarrow 1$ and Z becomes the BGS partition function.

Now let us suppose a continuous quantity \hat{x} within $0 < x < \infty$ replacing energy in the invariant distribution. Then Z may be given by the following integrations

$$Z = \int_0^\infty [1 - (1-q)\beta(x - \bar{x})]^{\frac{1}{1-q}} dx. \quad (33)$$

or

$$Z = \int_0^\infty [1 - (1 - q)\beta(x - \bar{x})]^\frac{q}{1-q} dx. \quad (34)$$

In this case, we should put $q > 1$ for Z to be calculated when x is large. The integration of Eq.(33) is always finite. But Eq.(34) needs $q < 2$ to be finite. If $q > 2$, the Z of Eq.(33) can be calculated while that of Eq.(34) diverges. This paradox naturally disappears for $q \rightarrow 1$.

3. The third problem concerns the calculation of expectation with Eq.(4) and the concomitant distribution Eq.(5). Usually, this calculation allows to establish a relation between, e.g., internal energy E and micro-state energies e_i through an $E - Z$ relationship ($U = -\frac{\partial}{\partial\beta} \ln Z$ within BGS). This is the crucial step in the statistical interpretation of thermodynamics. From Eq.(5), it is obvious that E calculation is self-referential : $E = O[p_i(E)]$ where $C[.]$ is certain mathematical operation. This calculus does not make sense for E because Eq.(5) is itself a relative distribution with respect to the internal energy E . So E may be arbitrary for a given spectrum e_i . To see this, we introduce the distribution function Eq.(5) into the expectation Eq.(4) to obtain :

$$\begin{aligned} E &= \frac{\sum_i p_i^q e_i}{Z^{1-q}} \\ &= \frac{1}{Z} \sum_i e_i [1 - (1 - q)\beta(e_i - E)]^\frac{q}{1-q} \\ &= -\frac{1}{Z} \left\{ \frac{\partial}{\partial\beta} \sum_i [1 - (1 - q)\beta(e_i - E)]^\frac{1}{1-q} - \sum_i E [1 - (1 - q)\beta(e_i - E)]^\frac{q}{1-q} \right\}. \end{aligned} \quad (35)$$

Considering Eqs.(20) and (21), we get

$$\begin{aligned} E &= -\frac{1}{Z} \left\{ \frac{\partial Z}{\partial\beta} - EZ \right\} \\ &= -\frac{1}{Z} \frac{\partial Z}{\partial\beta} + E \end{aligned} \quad (36)$$

which leads to, instead of the expected $U - Z$ relation,

$$\frac{\partial Z}{\partial\beta} = 0. \quad (37)$$

So no relation between macroscopic quantities and the correspondent microscopic quantities can be found from the expectation definition Eq.(4)

and arbitrary expectation is possible for any given states. The thermodynamics connection of this statistical mechanics becomes questionable even impossible.

4. In addition, Eq.(37) gives rise to another problem similar to the second one discussed above. Eq.(37) can be easily verified if we take the standard Z given by Eq.(6). But If we take the Z given by Eq.(20), we get the following relation

$$\sum_i^w (e_i - E) [1 - (1 - q)\beta(e_i - E)]^{\frac{2q-1}{1-q}} = 0 \quad (38)$$

or

$$E = \frac{\sum_i^w e_i [1 - (1 - q)\beta(e_i - E)]^{\frac{2q-1}{1-q}}}{\sum_i^w [1 - (1 - q)\beta(e_i - E)]^{\frac{2q-1}{1-q}}} = \frac{\sum_i^w e_i p_i^{2q-1}}{\sum_i^w p_i^{2q-1}}. \quad (39)$$

If we repeat the same reasoning with the Z of Eq.(20), we get

$$\begin{aligned} E &= \frac{\sum_i^w e_i p_i^q}{\sum_i^w p_i^q} \\ &= \frac{\sum_i^w e_i p_i^{2q-1}}{\sum_i^w p_i^{2q-1}} \\ &= \frac{\sum_i^w e_i p_i^{3q-2}}{\sum_i^w p_i^{3q-2}} \\ &= \frac{\sum_i^w e_i p_i^{4q-3}}{\sum_i^w p_i^{4q-3}} \\ &\dots = \frac{\sum_i^w e_i p_i^{nq-n+1}}{\sum_i^w p_i^{nq-n+1}}. \end{aligned} \quad (40)$$

which means $\sum_i^w (e_i - E) = 0$ or $E = \sum_i^w e_i / w$ if $q = \frac{n-1}{n}$ with $n = 1, 2, 3, \dots$ i.e., we are led to the microcanonical case. This seems contradictory to the generality of Eq.(21).

Above mathematical difficulties or peculiarities may lead to other difficulties for the theory and seriously harm the reliability of NSM with invariant distribution Eq.(5). The validity of escort probability and the resulted expectation in NSM also become questionable because it gives invariant distributions with respect to the expectation when applied to either Tsallis' or Rényi's entropy [12,15].

V. AN INVARIANCE RECOVERED WITH A GENERALIZED ENERGY SHIFT

Though Eq.(1) is not invariant with conventional energy shift ($e_i + C$), it can be invariant with a different energy shift we refer to as generalized energy translation.

We remember the product probability problem mentioned at the beginning of the previous section. The existence of thermodynamic equilibrium in an interacting system described by Tsallis entropy needs not only Eq.(23) and the factorization of total probability given by Eq.(22), also an energy pseudoadditivity similar to Eq.(23) and consistent with the product law of probability Eq.(22) [14,7], that is

$$e_{ij}(A + B) = e_i(A) + e_j(B) - (1 - q)\beta e_i(A)e_j(B). \quad (41)$$

and $p_{ij}[e_{ij}(A + B)] = p_i[e_i(A)]p_j[e_j(B)]$. Eq.(41) can be written in the conventional addition form with a generalized q -addition “ $+_q$ ”, i.e. $e_{ij}(A + B) = e_i(A) +_q e_j(B)$. Let q -exponential distribution Eq.(1) be noted by $p_i = \exp_q(-\beta e_i)$, we get

$$\begin{aligned} p_i(e_i +_q C) &= \frac{1}{Z(e + C)} \exp_q[-\beta(e_i +_q C)] \\ &= \frac{1}{Z(e)Z(C)} \exp_q(-\beta C) \exp_q(-\beta e_i) = \frac{1}{Z(e)} \exp_q(-\beta e_i) \\ &= p_i(e_i). \end{aligned} \quad (42)$$

where $Z(C) = \exp_q(-\beta C)$ and $Z(e) = \sum_i \exp_q(-\beta e_i)$. So the invariance of probability is recovered in a more general mathematical and physical context for systems with complex interactions or chaotic space time. A complete physical comprehension of this “nonlinear” energy shift certainly needs further investigation.

Here we would like to mention that, in this generalized context, we can construct a generalized arithmetic or algebra based on the q -exponential and its inverse function q -logarithm ($\ln_q(x) = \frac{x^{1-q}-1}{1-q}$) with, in addition to $+_q$, $-_q$, \times_q and \div_q (or $/_q$) corresponding respectively to the following arithmetic properties : $\exp_q(-_q x) = 1/\exp_q(x)$, $\ln_q(x \times_q y) = \ln_q(x) + \ln_q(y)$ and $\ln_q(1/_q y) = -\ln_q(y)$. This q -algebra is semi-classical due to its relationships containing simultaneously classical and q -operations, but it is useful for NSM and consistent with the complex physical circumstance. Considering the fact that the classical arithmetic is constructed on the basis of a *fragmented world*

with independent parts, we conjecture that generalizations are necessary for *the real messy world containing correlated, entangled and overlapped parts*. We have already seen the development of a κ -algebra based on a κ -statistics [16]. The above q -algebra is also a possible one for a different physical circumstance. Detailed discussion of this mathematical scenario will be given in another paper of ours.

VI. CONCLUSION

We have discussed some problems related to the energy translation invariance of probability distribution. It is shown that a probability distribution can be invariant only in the thermodynamic limit and only if there is no long term interaction or correlation and no relativistic effect is considered. So we believe that this invariance is not to be imposed to nonextensive statistics for complex systems containing long term interaction. Some theoretical peculiarities of the invariant distribution reveal that the thermodynamic connection of the NSM might be disturbed by this invariance. So the normalized expectation Eq.(4) becomes problematic for NSM. In addition, the distributions given by this expectation is numerically proved different from that predicted by the law of large numbers [17]. All these problems show that it is necessary to reconsider the role played by the escort probability in NSM. It was shown that the invariant property of probability distribution can exist with a generalized energy shift which is consistent with the factorization of joint probability (product law of probability) prescribed by the existence of thermodynamic equilibrium in correlated systems.

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